15 Multiple Integrals

15.1 Double Integrals over Rectangles

1. Recall that the definite integral of f (where f is a function of a single variable) from a to b is defined as: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x$, where $\Delta x = (b-a)/n$ (Usually this is arrived at by dividing the interval in subintervals etc)

2. For a function of two variables, you can graph it as a surface. Now suppose $f(x, y)$ is a function of two variables defined on a closed rectangle $R = [a, b] \times [c, d]$. Finding the volume of a solid S formed by (a) a function $f(x, y)$, (b) a region R below $f(x, y)$, and (c) the space between $f(x, y)$ and R; that is:
   $S = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} | 0 \leq z \leq f(x, y), (x, y) \in \text{the region } R\}$

3. in finding the volume, we use a method similar to the one of finding the area under a curve in one dimensional calc:

   - since R is rectangular, let $R = [a, b] \times [c, d]$
   - divide R into small rectangles R_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ that are Δx by Δy, where $\Delta x = \frac{b-a}{m}$ and $\Delta y = \frac{d-c}{n}$
   - choose $(x_{ij}^*, y_{ij}^*) \in R_{ij}$, where $R = \cup_{i,j} R_{ij}$
   - the volume is approximately the double Riemann sum: $V \approx \displaystyle\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$,

     where ΔA is the area of R_{ij} $(\Delta A = \Delta x \Delta y)$
   - as $m, n \to \infty$, the sum gives the exact volume:

   $$V = \iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

4. If $f(x, y) \geq 0$ on and inside R, one can interpret this as an approimation. of the volume under the surface $f(x, y)$: In finding the iterated integral, one can integrate with respect to y first (considering x to be a constant), and so we obtain a function of x only. This can be viewed as the area of a cross-section of the desired volume in the plane through x, perpendicular to the x-axis. Integrating this with respect to y gives the volume under the curve.

5. one can approximate the volume using the midpoint rule or the average value as we did for the area.

6. ways to solve the double integral coming up in the next section

15.2 Iterated Integrals over Rectangles

1. In finding the iterated integral $\iint\limits_R f(x,y)dA$, one can integrate with respect to y first (considering x to be a constant), obtaining a function of x only, and then evaluating this integral with respect to x. Fubini's theorem says that one can integrate with respect to x first, and then with respect to y in obtaining the same result under the conditions of the theorem.

2. In evaluating these integrals, use is made of a special case of Fubini's theorem (if the iterated integral exists): If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then: $\iint\limits_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

3. if $f(x,y) = g(x) \cdot h(y)$ (i.e. say $f(x,y) = x^3 e^y$, then $g(x) = x^3$ and $h(y) = e^y$), then $\int_c^d \int_a^b f(x,y)dxdy = \int_a^b g(x)dx \int_c^d h(y)dy$

15.3 Double Integrals over General Regions

1. If the region D is not a rectangular region, then we define

$$F(x,y) \quad = \quad \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{otherwise.} \end{cases}$$

and so $F(x,y)$ is defined over a rectangle now, and it contributes the same amount as $f(x,y)$. Then $\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA$

2. there are two types of regions:

   - Type I: D lies between the graphs of two continuous functions of x (on $[a,b]$), say $D = \{(x,y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. Then $$\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$
   - Type II: D lies between the graphs of two continuous functions of y (on $[c,d]$), say $D = \{(x,y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$. Then $$\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

3. properties: if D can be split, say $D = D_1 \cup D_2$, then the double integral over D is the sum of the double integrals over D_1 and D_2, respectively

4. $\displaystyle\iint\limits_D 1dA = A(D) = \text{area of } D$

5. if the function is bounded below and above on D, so is the integral (moreover, bounded in terms of the min and max values of $f(x,y)$): if $m \leq f(x,y) \leq M$, then
$$\iint\limits_D m\,dA \leq \iint\limits_D f(x,y)dA \leq \iint\limits_D M\,dA,$$
or equivalently
$$m \cdot A(D) \leq \iint\limits_D f(x,y)dA \leq M \cdot A(D).$$

15.4 Double Integrals in Polar Coordinates

1. a polar rectangle:
$$R = \{(r,\theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$
where $a, b \geq 0$ and $0 \leq \alpha, \beta \leq 2\pi$

2. then
$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\; r\; dr\; d\theta$$

3. in particular if $f(x,y)$ is continuous on the polar region (that is not necessarily a polar rectangle) $D = \{(r,\theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then
$$\iint\limits_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\; r\; dr\; d\theta$$

15.7 Triple Integrals

Triple integrals over rectangular boxes (the equivalent of a rectangle for double integrals)

1. we define triple integrals for functions of three variables, the same way we defined them for single and double integrals for functions of one and two variables, respectively. Let B be a rectangular box, then
$$\iiint\limits_B f(x,y,z)dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$
if the limit exists

2. In evaluating these integrals, we make use of a special case of Fubini's theorem (if the iterated integral exists): If f is continuous on the rectangular box $R = [a,b] \times [c,d] \times [r,s]$, then:

$$\iiint\limits_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz = \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx$$

or any other ordering

Let E be any general bounded region in 3 dimensions (i.e. a solid that is not necessarily a box)

1. We construct a larger box B that contains E and define the function F to be zero outside of E, and to match f over E. Then $\iiint\limits_E f(x,y,z)dV = \iiint\limits_B F(x,y,z)dV$, and this integral exists if f is continuous and the boundary of E is smooth (i.e. derivative is continuous and nonzero except maybe at the endpoints)
2. $\iiint\limits_E dV = V(E)$
3. there are three types of regions for E:

TYPE 1:

$$E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}.$$

Then the projection D of E on to the xy-plane could be of the two types as we had in Section 15.3

- D is Type I: y lies between the graphs of two continuous functions of x on D, then

$$E = \{(x,y,z) | a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}.$$

Then

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \Big(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\Big)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$$

- D is Type II: x lies between the graphs of two continuous functions of y on D, then

$$E = \{(x,y,z) | c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\}.$$

Then

$$\iiint\limits_E f(x,y,z)dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy$$

TYPE 2:

$$E = \{(x,y,z)|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}.$$

We then have (just as above)

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \Big(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\Big)dA$$

Also, D could be of type I or II as above.

TYPE 3:

$$E = \{(x,y,z)|(x,z) \in D, u_1(x,z) \le y \le u_2(x,z)\}.$$

We then have (just as above)

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \Big(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\Big)dA$$

Also, D could be of type I or II as above.

4. figuring out what type the integral is will not be as important as understanding the process of how to set up the integral–we'll do this in class

5. what is the meaning of the triple integral? Following the pattern with the integral and doubleintegral, the third one gives the hypervolume of a four dimensional object whose volume that we integrate over is E–a three dimensional object. Even better, one can find the

   - mass of an object by solving

     $$m = \iiint\limits_E \rho(x,y,z)dV,$$

     where $\rho(x,y,z)$ is the density function

   - moments about the three coordinate planes by solving

     $$M_{yz} = \iiint\limits_E x\rho(x,y,z)dV \; M_{xz} = \iiint\limits_E y\rho(x,y,z)dV \; M_{xy} = \iiint\limits_E z\rho(x,y,z)dV$$

     which also gives the center of mass $(x,y,z) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

   - and others

15.8 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates:

1. recall the cylindrical coordinates: r, θ, z $(r \geq 0, 0 \leq \theta \leq 2\pi)$ with $x = r\cos\theta, y = r\sin\theta$ and $z = z$ $(r^2 = x^2 + y^2, \tan\theta = \frac{y}{x})$

2. integrals over E whose projection D on the xy-plane is conveniently described in polar coordinates, say

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

with D (for $0 \leq \alpha \leq \beta \leq 2\pi$):

$$D = \{(r, \theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then the formula for triple integration in cylindrical coordinates is

$$\iiint_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; r\,dz\,dr\,d\theta$$

Spherical Coordinates:

1. help find integrals over a spherical edge (a chunk of the interior of the sphere, whose projection on the xy-plane is not an area that we know how to work with)

2. also, helps in solving integrals that are hard or impossible in rectangular coordinates (example 3 and the note after Example 3)

3. recall the spherical coordinates: ρ, θ, ϕ $(\rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi)$ with $x = \rho\sin\theta\cos\theta, y = \rho\sin\theta\sin\theta, z = \rho\cos\phi$ $(\rho = z^2 + y^2 + z^2)$

4. integrals over E whose projection D on the xy-plane is conveniently described in polar coordinates, say

$$E = \{(\rho, \theta, \phi)|a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

then the formula for triple integration in spherical coordinates is

$$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\; \rho^2\sin\phi\; d\rho\, d\theta\, d\phi,$$

where $a, b \geq 0; 0 \leq \alpha, \beta \leq 2\pi; c, d \geq 0$

15.9 Change of Variables in Multiple Integrals

1. we've seen before that sometimes it is easier to solve a double integral in polar coordinates than it is to solve it in rectangular ones. How do we transform an integral in one system of coordinates to a different one? Using a linear transformation T (a C^1 transformation), which is a continuous function from the first system of coordinates (like the rectangular one) to the second system (like the polar one), and this function needs to have continuous first order partials

2. if T is a one-to-one transformation to its range (i.e. it is a one-to-one correspondence (also called bijection) so that no two elements get mapped to the same element, and there is no element in the polar system that doesn't get hit) then there is an inverse transformation T^{-1} from the polar coordinate system to the rectangular one. What this says is that you can go back and forth between the two coordinate systems as needed. In doing so, we need to
   - map the region R to region S
   - find the nonzero Jacobian and use the change of variable formula for double integrals

3. when we map one region to another using a transformation T, we map boundaries to boundaries first, and then their interior.

4. recall that <u>the change of variable formula for integrals</u> is

$$\int_a^b f(x)dx = \int_c^d f(x(u))\, \frac{dx}{du}\, du$$

5. <u>The change of variable formula for double integrals:</u> If T is a one-to-one C^1 transformation (i.e. one-to-one except possibly on the boundary, continuous and with continuous partials) that maps a region S in the uv-plane onto a region R in the xy-plane, then

$$\iint\limits_R f(x,y)dA = \iint\limits_S f\Big(x(u,v), y(u,v)\Big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\, du\, dv$$

   where the <u>Jacobian</u> matrix of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0$$

6. note that the Jacobian **needs to be nonzero**.

7. try it to see how you get the double integral formula from rectangular coordinates to the polar coordinates

8. The change of variable formula for triple integrals: If T is a one-to-one C^1 transformation (i.e. one-to-one except possibly on the boundary, continuous and with continuous partials) that maps a region S in the uvw-space onto a region R in the xyz-space, then

$$\iiint_R f(x,y,z)dV = \iiint_S f\Big(x(u,v,w), y(u,v,w), z(u,v,w)\Big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$

where the Jacobian matrix of the transformation T given by $x = g(u,v,w)$, $y = h(u,v,w)$ and $z = k(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$$

9. note that the Jacobian **needs to be nonzero**.